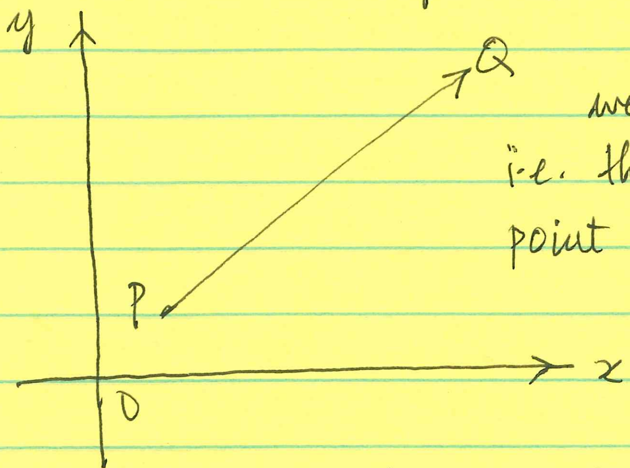
I. Vectors in a Plane

0° <u>Definition</u>: A vector in a plane (or space) is a directed line segment with an initial point P and a terminal pt. Q

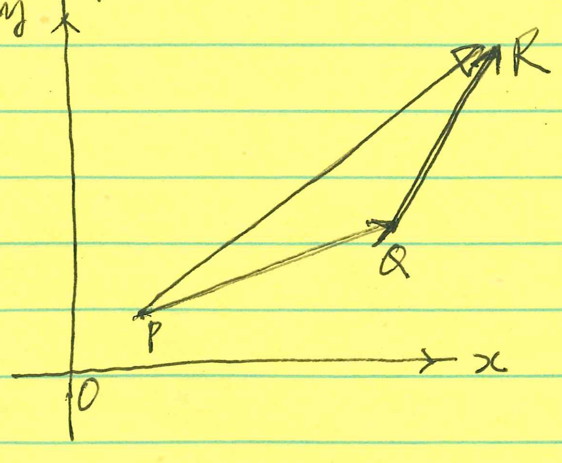


we denote the vector by $\overrightarrow{PQ}$ i.e. the vector pointing from the point P to the point Q.

<u>Two vectors are equal (regardless of where their initial pts. are) iff they have the same length (magnitude) as well as the same direction. We shall denote its length/magnitude by $|\overrightarrow{PQ}|$. (This simply tells you how long PQ is regardless of its direction).</u>

1° <u>Vector Addition (Geometric Approach)</u>:

<u>Motivation</u>: Suppose we visualize $\overrightarrow{PQ}$ as the displacement of an moving object from P to Q and $\overrightarrow{QR}$ as a further displacement of it from Q to R, then the position of the object would eventually end up at the pt. R (see picture).



The result would be like going from P to R. This motivates:

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} \leftarrow \text{the resultant vector of } \overrightarrow{PQ} + \overrightarrow{QR}.$$

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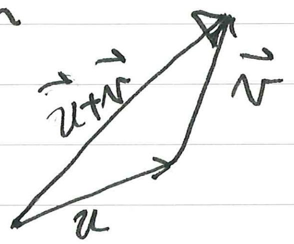
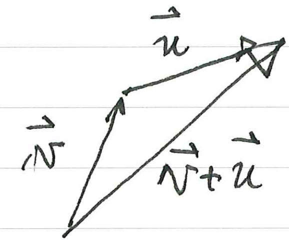
Generalizing this idea, we have two ways of vector addition:

2°(a) Δ-Law of vector addition

Two vectors $\vec{u}$ and $\vec{v}$ are being placed in a head-tail configuration



alternatively

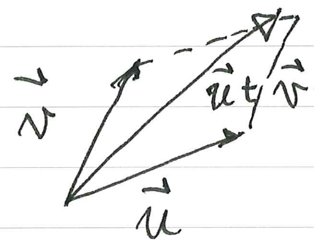


$\vec{u}+\vec{v}$ and $\vec{v}+\vec{u}$ have same length and direction, therefore

$\boxed{\vec{u}+\vec{v}=\vec{v}+\vec{u}}$ Commutative Law of vector addition.

2°(b) Parallel-Law of vector addition

$\vec{u}$ and $\vec{v}$ are in a tail-tail configuration



Complete the $\parallel$gm determined by $\vec{u}$ and $\vec{v}$, $\vec{u}+\vec{v}$ is given by the diagonal between $\vec{u}$ and $\vec{v}$, with its tail at the tail of the 2 vectors and the head at the opposite corner of the $\parallel$gm.

3° Scalar Multiplication of vectors:

Defn. A scalar c is just a real number.

Now that given any vector $\vec{v}$, we want to define $c\vec{v}$, there are 3 cases:

Case (i) $c=0$

$c\vec{v}=0\cdot\vec{v}=\vec{0}$ where $\vec{0}$ is known as the zero vector, one with zero length.

$0\vec{v}=\ \bullet =\vec{0}$, $|\vec{0}|=0$ ($|\ |$ denotes the length of a vector).

it degenerates into a single pt. a vector of length zero.

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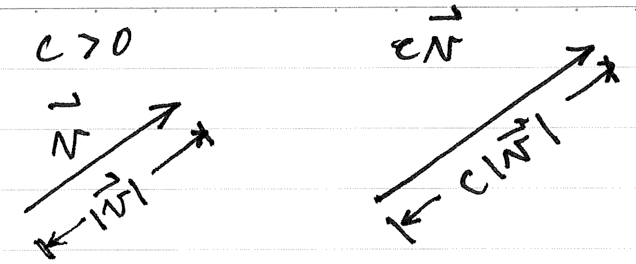
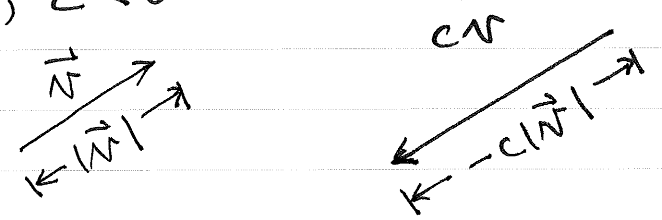
Case (ii) $c > 0$



$|c\vec{v}| = c|\vec{v}|$.

$c\vec{v}$ is in the same direction of $\vec{v}$, length stretched or compressed by a factor of c (depending on whether $c > 1$ or $0 < c < 1$).

Case (iii) $c < 0$

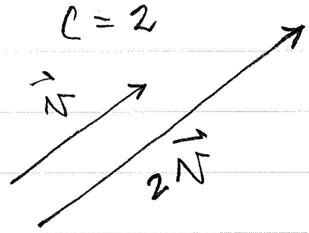


$|c\vec{v}| = -c|\vec{v}|$

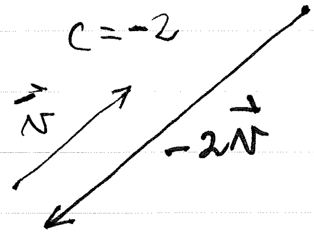
$c\vec{v}$ is in opposite direction of $\vec{v}$ with length stretched or compressed by a factor of $-c$ or $|c|$

In all cases, $\boxed{|c\vec{v}| = |c||\vec{v}|}$

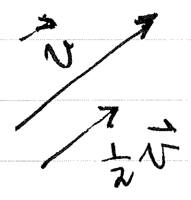
Ex. $c = 2$



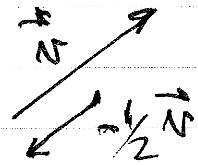
$c = -2$



Ex $c = \frac{1}{2}$

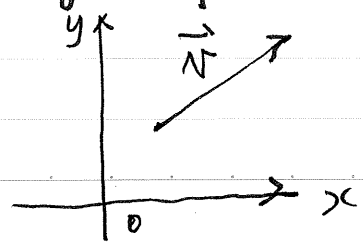


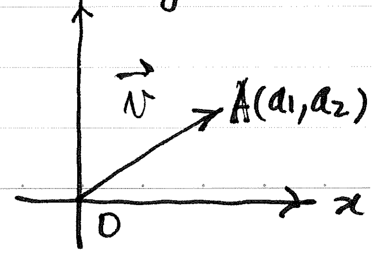
$c = -\frac{1}{2}$



4° <u>Position vector Representation of $\vec{v}$:</u>

Standard Configuration — We translate $\vec{v}$ so that its tail is at the origin of the co-ordinate system.





Suppose now the head of $\vec{v}$ is at the point $A(a_1, a_2)$, then

$$\vec{v} = \overrightarrow{OA}$$

where $\overrightarrow{OA}$ is known as the position vector pointing from the origin O to the point $A(a_1, a_2)$.

We denote $\vec{v}$ or $\overrightarrow{OA}$ by $\langle a_1, a_2 \rangle$

Definition: $\langle a_1, a_2 \rangle$ is known as the position vector representation of $\vec{v}$, a_1 and a_2 are known respectively as the <u>x-component</u> and the <u>y-component</u> ~~of~~ of $\vec{v}$.

<u>Remarks:</u>

(i) a_1 & a_2 are scalars.

(ii) $\langle a_1, a_2 \rangle \neq (a_1, a_2)$

(iii) $|\vec{v}| = \sqrt{a_1^2 + a_2^2}$ (Phythagorus Theorem).

5° <u>Algebraic approach on the basic operations of vectors</u>

Given $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{u} = \langle u_1, u_2 \rangle$ (i.e. in their standard representation).

We have:

(0) $\vec{v} = \vec{u}$ iff $v_1 = u_1$ & $v_2 = u_2$.

(i) $\vec{v} \pm \vec{u} = \langle v_1, v_2 \rangle \pm \langle u_1, u_2 \rangle = \langle v_1 \pm u_1, v_2 \pm u_2 \rangle$

i.e. Just $\pm$ <u>component-wise</u>.

(ii) $c\vec{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle$ (note that $|c\vec{v}| = \sqrt{(cv_1)^2 + (cv_2)^2}$
$= |c||\vec{v}|$ as expected)

i.e. multiply <u>component-wise</u>.

(iii) The zero vector is given by $\vec{0} = \langle 0, 0 \rangle$.

(iv) $-\vec{v} = \langle -v_1, -v_2 \rangle$ is known as the <u>Additive Inverse</u> (as $-\vec{v} + \vec{v}$
$= \vec{v} + -\vec{v}$
$= \vec{0}$).

Def. $\vec{u}$ and $\vec{v}$ are said to be parallel iff they are either in the same direction or in the opposite direction.

Theorem: $\vec{u}$ and $\vec{v}$ are $\parallel$ to each other iff they are scalar multiple of each other.

Pf: Exercise.

Ex $\langle 3,4\rangle$, $\langle -3,-4\rangle$, $\langle 1, 4/3\rangle$ and $\langle -12,-16\rangle$ are $\parallel$.

Ex. Given $\vec{u}=\langle 1,3\rangle$ & $\vec{v}=\langle 2,-4\rangle$, compute $\vec{u}+3\vec{v}$, $3\vec{u}-4\vec{v}$ and $|3\vec{u}-4\vec{v}|$.

Ex. Show that $\langle -4,-6\rangle$ and $\langle 2,5\rangle$ are not $\parallel$.

Pf: If they are $\parallel$, we have $\langle -4,-6\rangle = c\langle 2,5\rangle$ for some scalar multiple c.

$$\langle -4,-6\rangle = c\langle 2,5\rangle \Rightarrow \langle -4,-6\rangle = \langle 2c,5c\rangle$$

$$\Rightarrow 2c=-4,\quad 5c=-6$$

$$\Rightarrow c=-2,\quad c=-6/5 \quad \rightarrow\leftarrow \text{ Contradiction}$$

$\therefore$ they are not $\parallel$.

6° A summary on the basic properties of vector operations

Set $V_2 = \{\langle x,y\rangle \mid x,y\in\mathbb{R}\}$

which is the space of all vectors in the xy plane, then we have the following summary.

Thm. Given $\vec{u},\vec{v},\vec{w}\in V_2$ and $c,d\in\mathbb{R}$, we have:

(i) $\vec{u}\pm\vec{v} = \vec{v}\pm\vec{u}$ (Commutative property of $\pm$)

(ii) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ (Associative property of $+$).

(iii) $\vec{u}\pm\vec{0}=\vec{u}$ (zero vector addition, or $\vec{0}$ as the additive identity).

(iv) $\vec{u}+(-\vec{u})=\vec{0}$ ($-\vec{u}$ is the additive inverse of $\vec{u}$)

(v) $\begin{cases} c(\vec{u} \pm \vec{v}) = c\vec{u} \pm c\vec{v} \\ (c \pm d)\vec{u} = c\vec{u} \pm d\vec{u} \end{cases}$ (Distributive Properties)

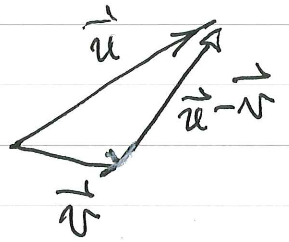
(vi) $\begin{cases} (1)\vec{u} = \vec{u} \\ (0)\vec{u} = \vec{0} \end{cases}$ (multiplication by 1 and 0).

(vii) $c(d\vec{u}) = d(c\vec{u}) = (cd)\vec{u}$ (associative property of scalar multiplication)

Remark: Geometric Interpretation of $\vec{u} - \vec{v}$ (Subtraction of vectors)

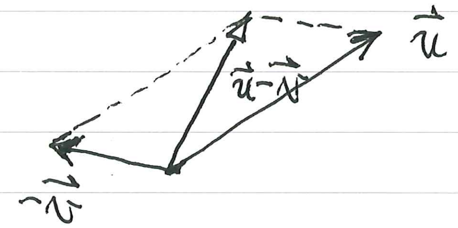
$\because$ $\vec{u} = (\vec{u} - \vec{v}) + \vec{v}$

$\therefore$ $\vec{u} - \vec{v}$ is the vector we add to $\vec{v}$ in order to get $\vec{u}$



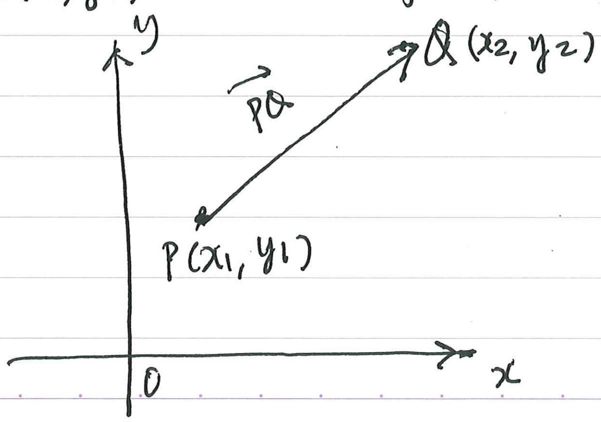
put $\vec{u}$ and $\vec{v}$ in a tail-tail configuration.

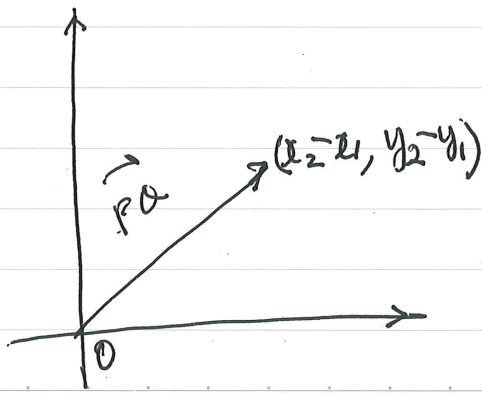
alternatively, $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$



7° Finding the position vector representation for $\vec{PQ}$

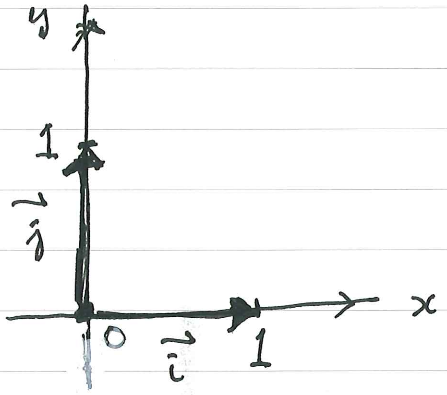
Consider $P(x_1, y_1)$ and $Q(x_2, y_2)$, $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$





8° Standard Vectors $\vec{i}$ and $\vec{j}$

We define $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$



Then, any $\vec{v} = \langle v_1, v_2 \rangle \in V_2$ could be expressed in the form,

$$\vec{v} = v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle$$
$$= v_1 \vec{i} + v_2 \vec{j} \leftarrow \text{a linear combination of } \vec{i} \text{ and } \vec{j}.$$

$v_1, v_2 \in \mathbb{R}$ are respectively known as the x-component and the y-component of $\vec{v}$.

Definition: We define $\{\vec{i}, \vec{j}\}$ to be a basis of V_2

Remark: In general, given any pair of vectors in V_2, $\{\vec{u_1}, \vec{u_2}\}$, if for any $\vec{v} \in V_2$ we could express $\vec{v}$ as a linear combination of $\vec{u_1}$ and $\vec{u_2}$ i.e.

$$\vec{v} = c_1 \vec{u_1} + c_2 \vec{u_2} \quad \text{for some } c_1, c_2 \in \mathbb{R},$$

then $\{\vec{u_1}, \vec{u_2}\}$ is said to be a basis of V_2.

9° Direction vector of $\vec{v}$

Definition: $\vec{u} \in V_2$ is said to be an unit vector if $|\vec{u}| = 1$

Definition: Given $\vec{v} \in V_2$ such that $\vec{v} \neq \vec{0}$, then the direction vector of $\vec{v}$ is defined to be $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ (normalization of $\vec{v}$)

i.e. The direction vector of $\vec{v}$ is the unit vector that points in the same direction as $\vec{v}$. Hence, we could always express $\vec{v}$ by

$\boxed{\vec{v} = |\vec{v}|\vec{u}}$ where u is the direction vector of v

↑ Direction vector representation of $\vec{v}$.

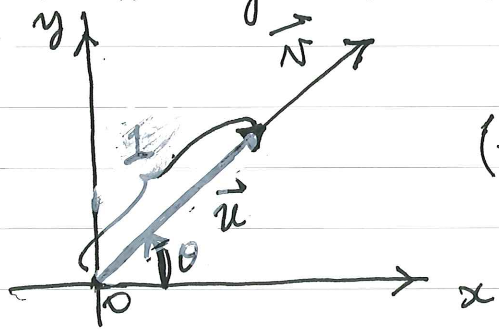
Ex. Express $\vec{v} = \langle 3, -4 \rangle$ in terms of its magnitude or length and its direction vector.

direction vector of $\vec{v}$ is given by $\vec{u} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$ $(|\vec{v}| = 5)$

$\therefore \vec{v} = |\vec{v}|\vec{u} = 5\langle \frac{3}{5}, \frac{-4}{5} \rangle$

Remarks (i) Advantage of this representation is, we could tell the length and the direction of $\vec{v}$ immediately.

(ii) Let θ be the angle $\vec{v}$ makes with the +ve x-axis,

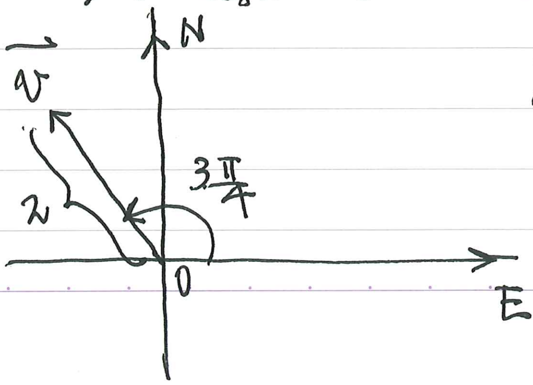


(The case when $|\vec{v}| > 1$)

Then $\vec{u} = \langle \cos\theta, \sin\theta \rangle$, $\vec{v} = |\vec{v}|\langle \cos\theta, \sin\theta \rangle$

Ex. Consider $\vec{v} = \langle -\sqrt{2}, \sqrt{2} \rangle$ then $|\vec{v}| = 2$

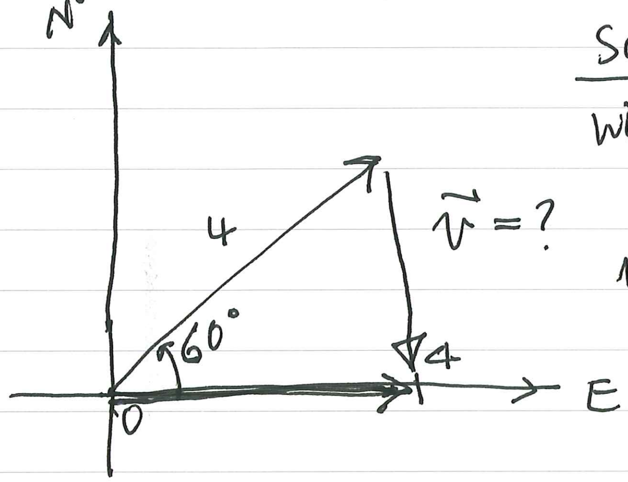
$\Rightarrow$ direction vector $\vec{u} = \langle \frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$



$\vec{v} = 2\langle \cos\frac{3\pi}{4}, \sin\frac{3\pi}{4} \rangle$

$\therefore$ length of $\vec{v}$ $= 2$, in the direction 45° N of W //

Ex. The water current of a ocean is 4 knots/hour 60° N of E. How would you steer your boat so that you end up traveling with a velocity of 4 knots/hour but in the East direction instead.



<u>Solution:</u> Let $\vec{v}$ be the velocity with which you steer your boat.

$\vec{v} = ?$

$$\text{water current} = 4\langle \cos 60°, \sin 60° \rangle$$
$$= 4\langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle$$
$$= \langle 2, 2\sqrt{3} \rangle$$

Resultant velocity $= \langle 4, 0 \rangle$

We need, $\langle 2, 2\sqrt{3} \rangle + \langle v_1, v_2 \rangle = \langle 4, 0 \rangle$

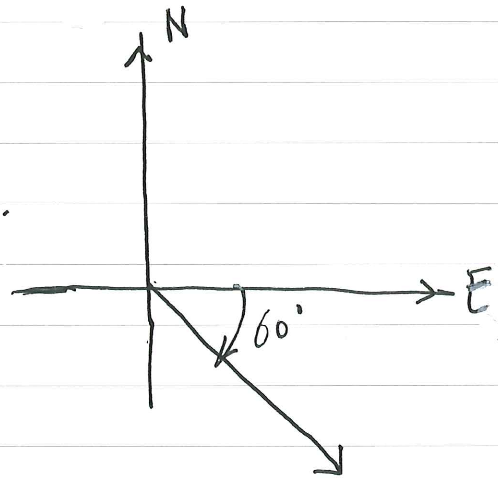
$$\Rightarrow v_1 + 2 = 4, \quad 2\sqrt{3} + v_2 = 0$$

$$\Rightarrow v_1 = 2, \quad v_2 = -2\sqrt{3}$$

$\therefore \vec{v} = \langle 2, -2\sqrt{3} \rangle, \quad |\vec{v}| = \sqrt{4+12} = 4$

$$\Rightarrow \vec{v} = 4\langle \tfrac{1}{2}, -\tfrac{\sqrt{3}}{2} \rangle$$

$\therefore$ 4 knots/hour, 60° S of E //.



Ex $\{\vec{i}, \vec{j}\}$ is not the only basis of V_2, indeed we have the following result.

Th^m. Let $\{\vec{a} = \langle a_1, a_2\rangle, \vec{b} = \langle b_1, b_2\rangle\}$ be any pair of non-zero vectors which are not parallel to each other, then $\{\vec{a}, \vec{b}\}$ forms a basis for V_2.

Pf: It suffices to prove for any $\vec{v} = \langle v_1, v_2\rangle$, there exists $x, y \in \mathbb{R}$ such that

$$\vec{v} = x\vec{a} + y\vec{b}$$

i.e.
$$\langle v_1, v_2\rangle = x\langle a_1, a_2\rangle + y\langle b_1, b_2\rangle$$
$$\langle v_1, v_2\rangle = \langle a_1x + b_1y, a_2x + b_2y\rangle$$

It boils down to solving
$$\begin{cases} a_1x + b_1y = v_1 & \text{---------- (i)} \\ a_2x + b_2y = v_2 & \text{---------- (ii)} \end{cases}$$

Since $\langle b_1, b_2\rangle \neq \langle 0, 0\rangle$, without loss of generality, we assume $b_1 \neq 0$

$$\Rightarrow y = \frac{v_1 - a_1x}{b_1}$$

substituting into (ii), $$a_2x + \frac{b_2v_1 - a_1b_2x}{b_1} = v_2$$

$$\Rightarrow \left(a_2 - \frac{a_1b_2}{b_1}\right)x = \frac{b_1v_2 - b_2v_1}{b_1}$$

But $a_2 - \frac{a_1b_2}{b_1} \neq 0$ otherwise we have $a_1 \neq 0$ and $\frac{a_2}{a_1} = \frac{b_2}{b_1}$ implying $\vec{a} \parallel \vec{b}$ and we have a contradiction.

Hence the theorem is proven //.

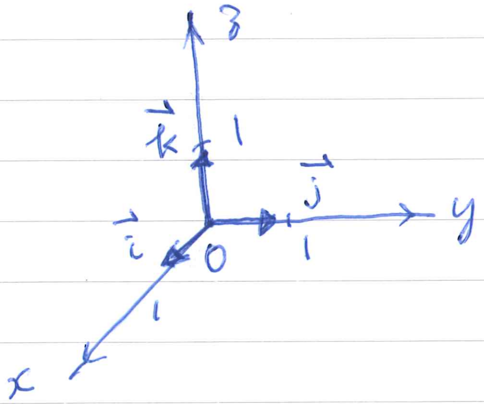
Date

II. Vectors in Space

Defn. $V_3 = \{ \vec{v} = \langle v_1, v_2, v_3 \rangle \mid v_1, v_2, v_3 \in \mathbb{R} \}$

Define $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$ and $\vec{k} = \langle 0,0,1 \rangle$

Everything from V_2 could be carried over.



$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

$\{\vec{i}, \vec{j}, \vec{k}\}$ forms a basis for V_3.

Ex Given $\vec{v} = \langle 4,3,7 \rangle$ express $\vec{v}$ in terms of its direction vector $\vec{u}$.

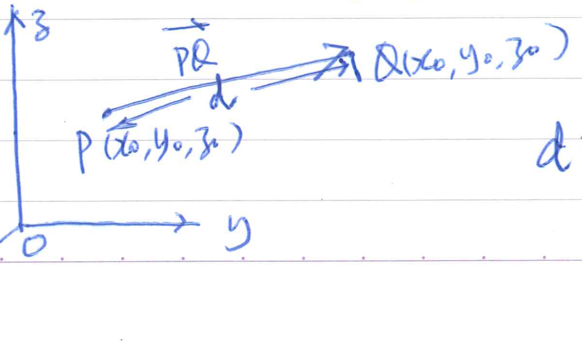
Solution: $\vec{v} = |\vec{v}|\, \vec{u}$ where $|\vec{v}| = \sqrt{4^2+3^2+7^2} = \sqrt{74}$

and $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \left\langle \dfrac{4}{\sqrt{74}}, \dfrac{3}{\sqrt{74}}, \dfrac{7}{\sqrt{74}} \right\rangle$

$$\therefore \vec{v} = \underset{\substack{\uparrow \\ \text{magnitude} \\ \text{of } \vec{v}}}{\sqrt{74}} \underbrace{\left\langle \frac{4}{\sqrt{74}}, \frac{3}{\sqrt{74}}, \frac{3}{\sqrt{74}} \right\rangle}_{\text{direction vector of } \vec{v}}$$

Distance Formula & Equation of Sphere

Thm. Given $P(x_0, y_0, z_0)$ & $Q(x_1, y_1, z_1)$ in space,

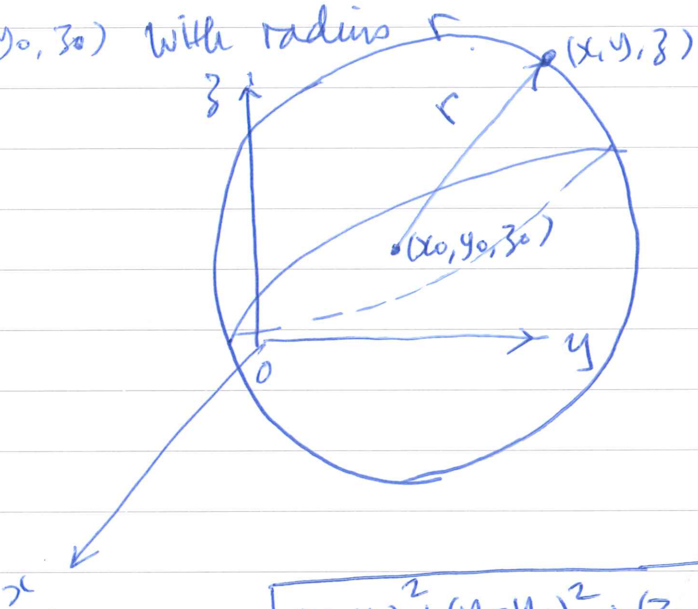


$$d = \sqrt{(x_1-x_0)^2 + (y_1-y_0)^2 + (z_1-z_0)^2}$$

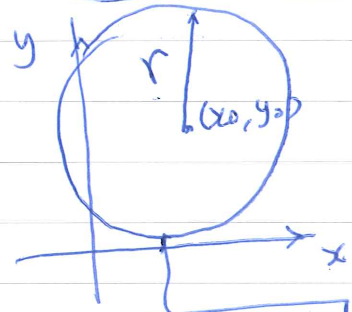
$\overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

$d = |\overrightarrow{PQ}| = \sqrt{(x_1-x_0)^2 + (y_1-y_0)^2 + (z_1-z_0)^2}$ //.

Let now (x, y, z) be any point on a sphere centered at (x_0, y_0, z_0) with radius r



Rmk. Eqn. of circle



$\boxed{(x-x_0)^2 + (y-y_0)^2 = r^2}$

We have $\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = r$,

alternatively $\boxed{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2}$

Ex Identify the sphere whose equation is

$$x^2 + x + y^2 - 2y + z^2 - 1 = 0$$

Ex. Determine whether $P(2,3,1)$, $Q(0,4,2)$ and $R(4,1,4)$ are co-linear.

Solution: $\overrightarrow{PQ} = \langle 0-2, 4-3, 2-1 \rangle = \langle -2, 1, 1 \rangle$

$\overrightarrow{QR} = \langle 4, -3, 2 \rangle$

But $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are not $\parallel$ since $\frac{4}{-2} \neq \frac{-3}{1}$ (or $\frac{4}{-2} \neq \frac{2}{1}$)

Hence not co-linear //.

Ex. The thrust of an airplane's engine could produce a speed of 600 mph in still air. The plane is steered in the direction of $\langle 2,2,1\rangle$ and the velocity of wind is $\langle 10,-20,0\rangle$. Find the velocity of plane with respect to the ground and its speed (i.e. how fast the plane is flying regardless of its direction).

Solution:

Basic Principle: Actual velocity or velocity with respect to the ground
= velocity of plane in still air + wind velocity.

Direction vector of $\langle 2,2,1\rangle = \dfrac{\langle 2,2,1\rangle}{|\langle 2,2,1\rangle|} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle$

$\therefore$ velocity of plane in still air $= 600\left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle = \langle 400, 400, 200\rangle$

velocity of wind $= \langle 10,-20,0\rangle$

$\therefore$ Actual velocity of plane $= \langle 400,400,200\rangle + \langle 10,-20,0\rangle$
$= \langle 410, 380, 200\rangle$ //

Speed of plane $= |\langle 410, 380, 200\rangle| = \sqrt{410^2+380^2+200^2}$ //

Dot Product / Inner Product / Scalar Product

Definition: Given $\vec{u} = \langle u_1,u_2,u_3\rangle$ and $\vec{v} = \langle v_1,v_2,v_3\rangle$, the inner product of $\vec{u}$ and $\vec{v}$ is defined by

$$\boxed{\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3}$$

Thm (Basic Properties of Dot Product)

(i) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ (Commutative property)

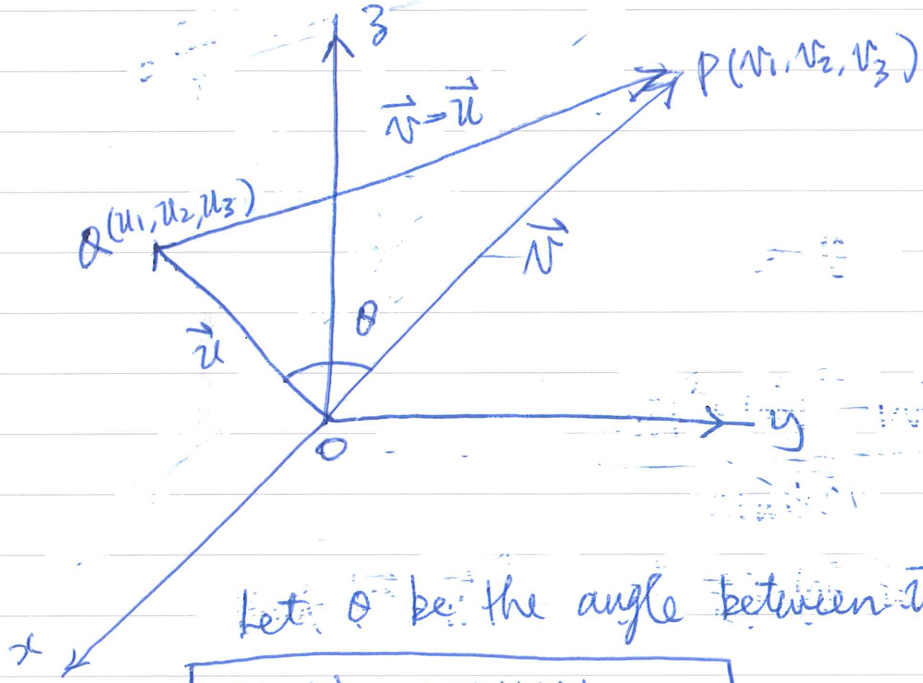
(ii) $\vec{u}\cdot(\vec{v}\pm\vec{w}) = \vec{u}\cdot\vec{v} \pm \vec{u}\cdot\vec{w}$ (Distributive Property)

(iii) $\vec{u}\cdot\vec{u} = |\vec{u}|^2$ or $|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}}$

Pf: Exercise.

Thm. (Geometric Meaning of dot product)

Given $\vec{u}$ and $\vec{v}$ in a tail-tail configuration



Let θ be the angle between $\vec{u}$ and $\vec{v}$, then

$$\boxed{\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta}$$

Pf:

$$\cos\theta = \frac{|\vec{u}|^2 + |\vec{v}|^2 - |\vec{v}-\vec{u}|^2}{2|\vec{u}||\vec{v}|}$$

But $|\vec{u}-\vec{v}| = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$

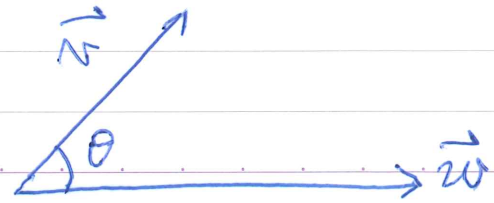
$$\Rightarrow \cos\theta = \frac{\cancel{|\vec{u}|^2} + \cancel{|\vec{v}|^2} - (\cancel{|\vec{u}|^2} + 2\vec{u}\cdot\vec{v} + \cancel{|\vec{v}|^2})}{2|\vec{u}||\vec{v}|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

whence the result //

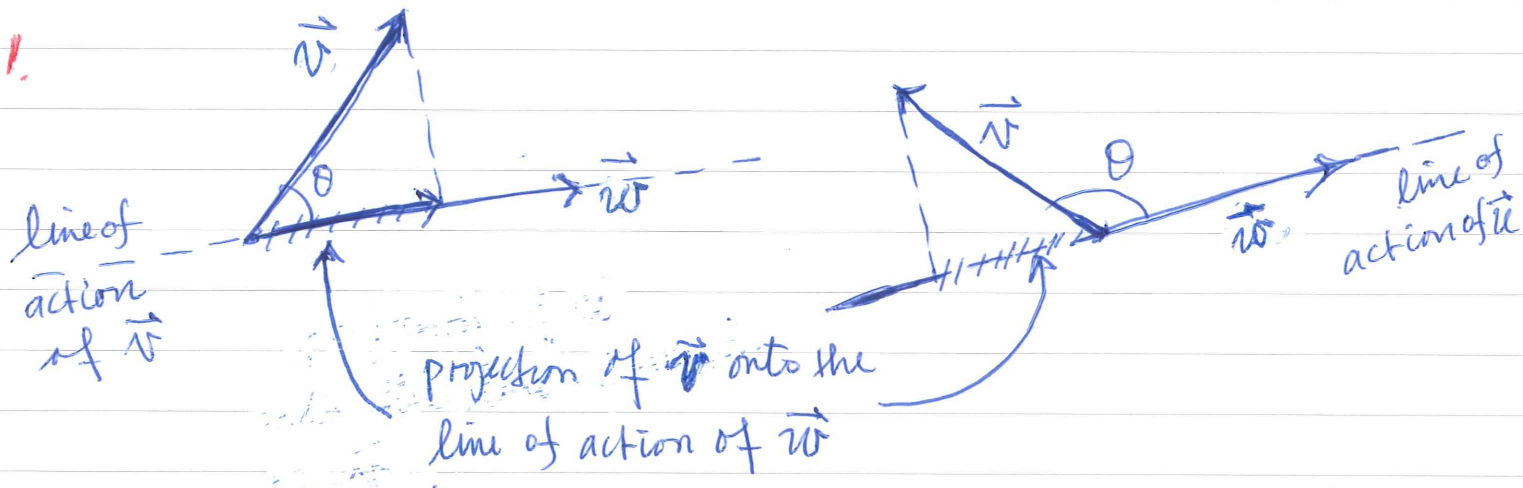
Corollary: $\vec{u} \perp \vec{v}$ iff $\vec{u}\cdot\vec{v} = 0$.

Component of a vector along another one

Defn. Let $\vec{v}$ and $\vec{w}$ be in a tail-tail configuration with an included angle θ. $(0 \le \theta \le \pi)$, the component of $\vec{v}$ along $\vec{w}$ is defined by



$$\boxed{\operatorname{Comp}_{\vec{w}}\vec{v} = |\vec{v}|\cos\theta}$$



Remarks (i) For $0 \leq \theta \leq \pi/2$, $\text{Comp}_{\vec w}\vec v \geq 0$ because the projection is on the same side as $\vec w$ and for $\pi/2 \leq \theta \leq \pi$, $\text{Comp}_{\vec w}\vec v \leq 0$ because the projection of $\vec w$ on the opposite side of $\vec w$.

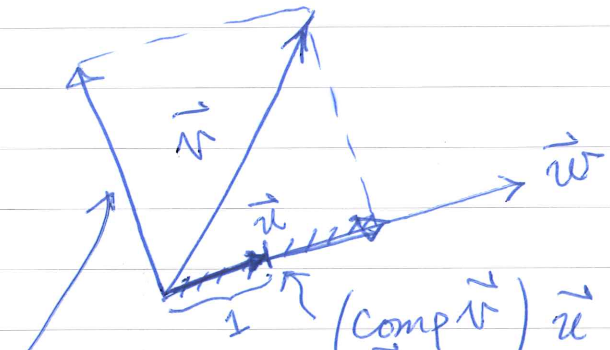
(ii)
$$\text{Comp}_{\vec w}\vec v = \frac{\vec v \cdot \vec w}{|\vec w|} \quad (\because \vec v \cdot \vec w = |\vec v||\vec w|\cos\theta$$
$$= \vec v \cdot \vec u \quad \text{where } \vec u = \frac{\vec w}{|\vec w|} \text{ is the direction vector of } \vec w.$$

(iv) $\text{Comp}_{\vec w}\vec v$ is the generalization of the x, y and z component of $\vec v$ along the vectors $\vec i, \vec j, \vec k$. Thus, for $\vec v = \langle v_1, v_2, v_3\rangle$,

$$\text{Comp}_{\vec i}\vec v = \langle v_1, v_2, v_3\rangle \cdot \langle 1,0,0\rangle = v_1 \quad (x\text{-component of } \vec v)$$
$$\text{Comp}_{\vec j}\vec v = v_2 \quad (y\text{-component of } \vec v)$$
$$\text{Comp}_{\vec k}\vec v = v_3 \quad (z\text{-component of } \vec v).$$

Date

1\. <u>Remark</u>: Resolution of vectors — $\vec{v}$ could always be decomposed or resolved into a sum of 2 vectors, one along $\vec{w}$ and the other one $\perp$ to $\vec{w}$ by $\parallel$gm law of vector addition



$\left(\text{comp}_{\vec{w}}\vec{v}\right)\vec{u} = (\vec{v}\cdot\vec{u})\vec{u}$ where $\vec{u}$ is the direction vector of $\vec{w}$

$= \left(\dfrac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\right)\vec{w}$

$\vec{v} - (\vec{v}\cdot\vec{u})\vec{u}$

5\. Thus, $\boxed{\vec{v} = (\vec{v}\cdot\vec{u})\vec{u} + \left(\vec{v} - (\vec{v}\cdot\vec{u})\vec{u}\right)}$

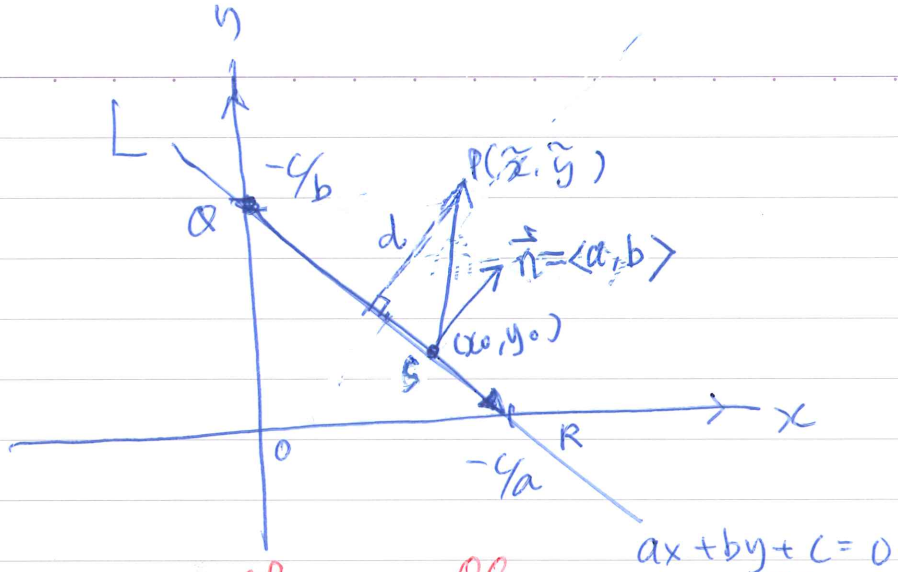
<u>Ex</u>. Application in Co-ordinate Geometry

Let $ax+by+c=0$ be the equation of a straight line L in the plane and let $P(\tilde{x},\tilde{y})$ be any point in the plane. Prove that d, the $\perp$ distance of P to L is given by

10\. $$d = \frac{|a\tilde{x}+b\tilde{y}+c|}{\sqrt{a^2+b^2}}$$

<u>Pf</u>: Wlog, assuming L is neither | nor — (otherwise would be trivial). In this case, $-c/a$ and $-c/b$ are respectively

13\. the x and y intercept of L.



OR OQ QR

$\overrightarrow{QR} = \langle -c/a, 0 \rangle - \langle 0, -c/b \rangle = \langle -c/a, c/b \rangle \parallel L$

It is trivial to see $\vec{n} = \langle a, b \rangle \perp L$ ($\because \vec{n} \cdot \overrightarrow{QR} = 0$)

Now that let $S(x_0, y_0)$ be any pt. on L

$\overrightarrow{SP} = \langle \tilde{x} - x_0, \tilde{y} - y_0 \rangle$, $d = \left| \operatorname{Comp}_{\vec{n}} \overrightarrow{SP} \right|$

$$\operatorname{Comp}_{\vec{n}} \overrightarrow{SP} = \frac{\overrightarrow{SP} \cdot \vec{n}}{|\vec{n}|} = \langle \tilde{x} - x_0, \tilde{y} - y_0 \rangle \cdot \frac{\langle a, b \rangle}{\sqrt{a^2 + b^2}}$$

$$= \frac{a\tilde{x} - ax_0 + b\tilde{y} - by_0}{\sqrt{a^2 + b^2}}$$

$$= \frac{a\tilde{x} + b\tilde{y} + c}{\sqrt{a^2 + y^2}}$$

$$\Rightarrow d = \frac{|a\tilde{x} + b\tilde{y} + \tilde{c}|}{\sqrt{a^2 + b^2}} \quad //$$

Cross Product

Def'n. Given $\vec{u} = \langle a_1, b_1, c_1 \rangle$ and $\vec{v} = \langle a_2, b_2, c_2 \rangle$, we define the cross product $\vec{u} \times \vec{v}$ by

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \vec{k}$$

$$= (b_1c_2 - c_1b_2)\vec{i} - (a_1c_2 - c_1a_2)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

$$= \langle b_1c_2 - c_1b_2,\ c_1a_2 - a_1c_2,\ a_1b_2 - a_2b_1 \rangle$$

Short Cut

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ a_1 & b_1 \\ a_2 & b_2 \end{matrix}$$

$$= (b_1c_2\vec{i} + c_1a_2\vec{j} + a_1b_2\vec{k}) - (a_2b_1\vec{k} + b_2c_1\vec{i} + c_2a_1\vec{j})$$

$$= \langle b_1c_2 - c_1b_2,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1 \rangle \ //$$

Ex

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 1 & 2 \\ 4 & 5 \end{matrix}$$

$$= 12\vec{i} + 12\vec{j} + 5\vec{k} - 8\vec{k} - 15\vec{i} - 6\vec{j}$$

$$= -3\vec{i} + 6\vec{j} - 3\vec{k}$$

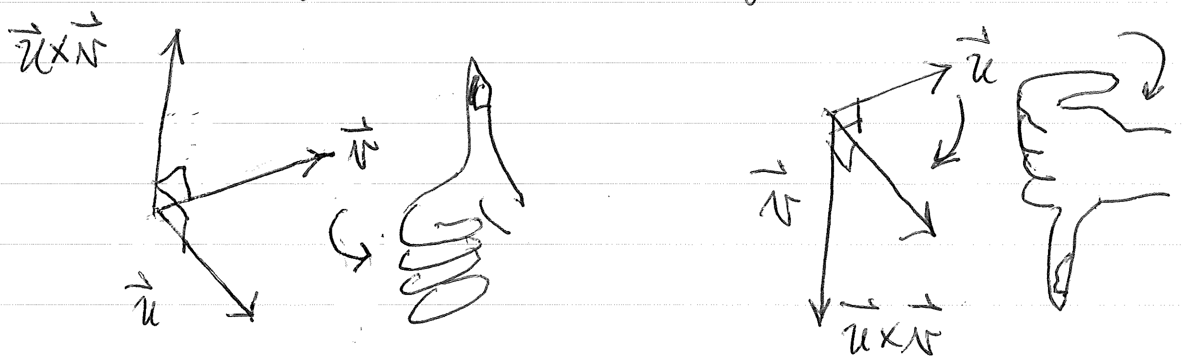
$$= \langle -3, 6, -3 \rangle \ //$$

1. <u>Geometric Meaning of $\vec{u}\times\vec{v}$</u>

$\vec{u}\times\vec{v}$ is a vector whose magnitude and direction are determined by the following: Let $\vec{u}$ and $\vec{v}$ be in a tail-tail configuration.

(i) $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$, $0\le\theta\le\pi$

(ii) Direction of $\vec{u}\times\vec{v}$ is determined by the <u>right hand rule</u>.

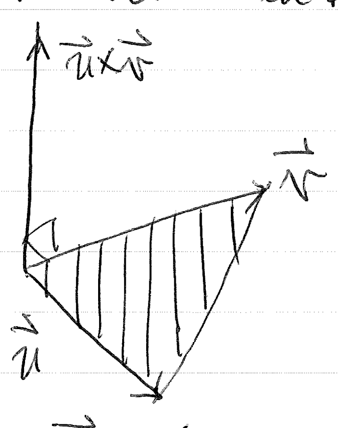


Curl your fingers in the direction from $\vec{u}$ to $\vec{v}$, then the thumb would turn into the direction of $\vec{u}\times\vec{v}$ which is $\perp$ to $\vec{u}$ and $\vec{v}$.

<u>Corollaries (Geometric applications):</u>

(i) $\vec{u}\times\vec{v}=\vec{0}$ iff $\vec{u}\parallel\vec{v}$.

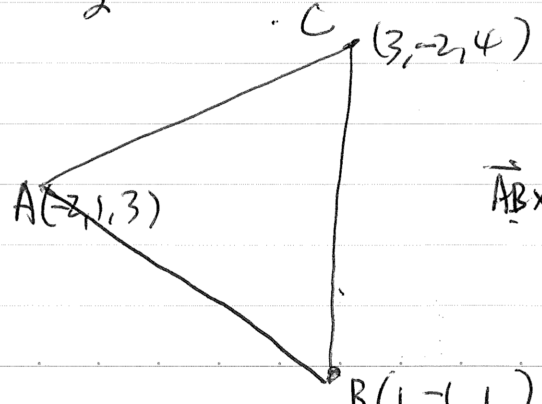
(ii) Let $\vec{u}$ and $\vec{v}$ be tail-tail, the Δ contained between $\vec{u}$ and $\vec{v}$ is given by: $\Delta = \frac{1}{2}|\vec{u}\times\vec{v}|$



Pf:

$$\Delta = \frac{1}{2}|\vec{u}||\vec{v}|\sin\theta$$
$$= \frac{1}{2}|\vec{u}\times\vec{v}|$$

Ex.



$\vec{AC} = \langle 5,-3,1\rangle$, $\vec{AB}=\langle 3,-2,-2\rangle$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -2 \\ 5 & -3 & 1 \end{vmatrix} = -8\vec{i}-13\vec{j}+\vec{k}$$

$$\triangle ABC = \frac{1}{2}\sqrt{64+169+1} = \frac{\sqrt{234}}{2} \,//$$

Thm. (Basic Properties of Cross Product)

(i) $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ (Non-commutative Property)

(ii) $\forall c\in\mathbb{R}$, $c\vec{u}\times\vec{v} = \vec{u}\times c\vec{v} = c(\vec{u}\times\vec{v})$ (Assoc Property for scalar multiplication).

(iii) $(\vec{u}\pm\vec{v})\times\vec{w} = \vec{u}\times\vec{w} \pm \vec{v}\times\vec{w}$

(iv) $\vec{w}\times(\vec{u}\pm\vec{v}) = \vec{w}\times\vec{u} \pm \vec{w}\times\vec{v}$

$\Big\}$ (Distributive Property)

(v) $\vec{i}, \vec{j}, \vec{k}$ satisfy a "cyclic situation"

$$\begin{tikzcd} & \vec{k} \arrow[dl] & \\ \vec{i} \arrow[rr] & & \vec{j} \arrow[ul] \end{tikzcd} \Leftrightarrow \vec{i}\times\vec{j}=\vec{k},\ \vec{j}\times\vec{k}=\vec{i},\ \vec{k}\times\vec{i}=\vec{j}.$$

<u>Scalar Triple Product</u>

Defn Given $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$ and $\vec{c} = \langle c_1, c_2, c_3\rangle$, we define the scalar-triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ by $\vec{a}\cdot(\vec{b}\times\vec{c})$

Thm $$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Pf: Since

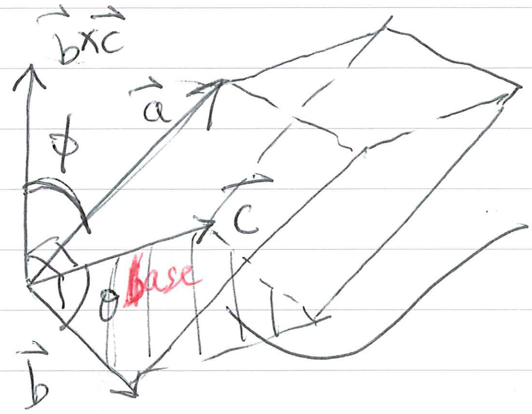
$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \left\langle \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right\rangle ;$$

therefore $$\vec{a}\cdot(\vec{b}\times\vec{c}) = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

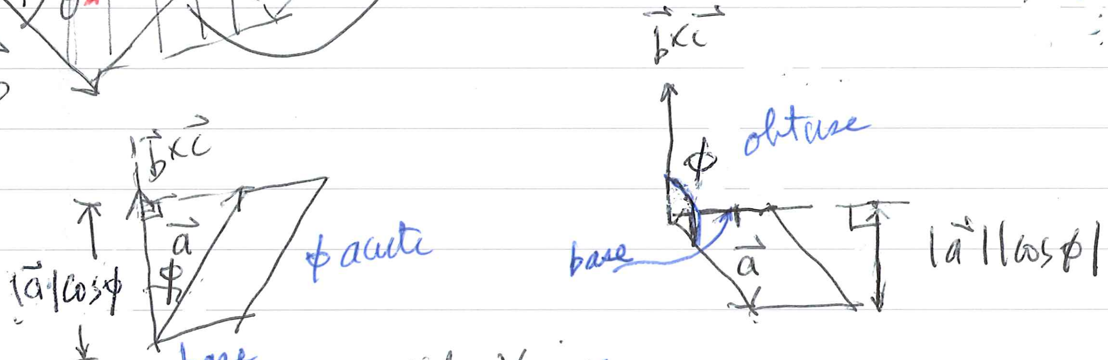
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} //$$

Remarks:- (i) $|\vec{a}\cdot(\vec{b}\times\vec{c})|$ = volume of ||pipe determined by $\vec{a}$, $\vec{b}$ & $\vec{c}$



$$|\vec{a}\cdot(\vec{b}\times\vec{c})| = \underbrace{|\vec{b}\times\vec{c}|}_{\text{base area of ||gm}}\underbrace{|\vec{a}|\cos\phi|}_{\text{height of ||pipe}}$$

$|\vec{b}\times\vec{c}|$ (base area of ||gm). height of ||pipe

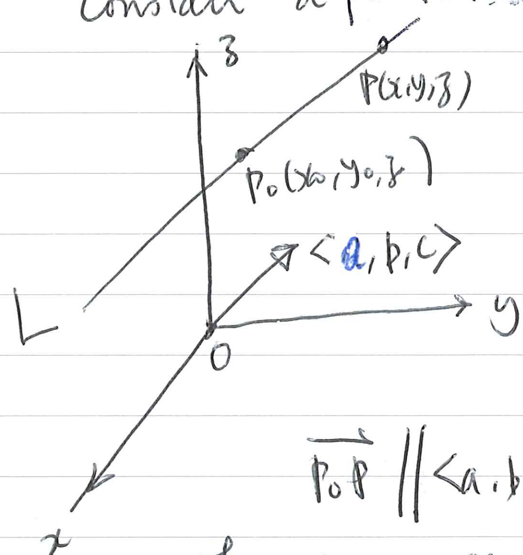


Side Views

(ii) $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar iff $\vec{a}\cdot(\vec{b}\times\vec{c})=0$.

III. Lines and Planes in Space.

Consider a pt. $P_0(x_0, y_0, z_0)$ a line L passes through, let $\langle a,b,c\rangle$ be any vector which is || to L.



Consider now any other pt. (x,y,z) on L. Then

$$\overrightarrow{P_0P} = \langle x-x_0, y-y_0, z-z_0\rangle$$

$$\overrightarrow{P_0P} \parallel \langle a,b,c\rangle \Rightarrow \langle x-x_0, y-y_0, z-z_0\rangle = t\langle a,b,c\rangle$$

for some scalar multiple t. Thus, by equating the x, y and z components, we have

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \quad -\infty < t < \infty.$$

This is the <u>parametric form or representation</u> of L.

Assuming now $a, b, c > 0$, then we have

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}} \quad (=t)$$

We call this the <u>symmetric form/representation of L.</u>

Remark: In particular if some of the a, b, c is zero, e.g. if $a=0$, we have

$$x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

Occasionally, we could also abuse our notation and write down the symmetric form of L as,

$$\frac{x-x_0}{0} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

with the understanding that " $\frac{x-x_0}{0} =$ " really means $\boxed{x = x_0}$

<u>KEY</u>: A line L is uniquely determined once a point P_0 on L and a vector $\parallel$ to L are given. Alternatively, L is uniquely determined once two distinct points P_0 and P_1 on L are given.

Ex. Find equations of a straight line L passing through $P(3,1,2)$ & $Q(4,3,1)$ in parametric and symmetric form.

<u>Solution</u>:

We may take $P(3,1,2)$ as our reference pt. (i.e. our (x_0, y_0, z_0)) and take $\overrightarrow{PQ} = \langle 1, 2, -1 \rangle$ to be $\parallel$ to L.

Then we have $\quad L: x = 3+t, \; y = 1+2t, \; z = 2-t$ (parametric form) //

$$\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{-1} \quad \text{(symmetric form)} \; //$$

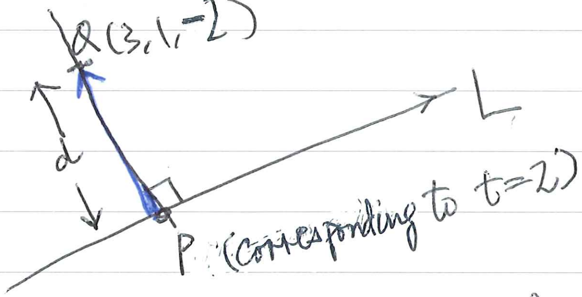
Ex Show that $A(3,1,0)$, $B(2,2,2)$ and $C(0,4,6)$ are co-linear (on a straight line)

Solution: $\overrightarrow{AB} = \langle -1, 1, 2\rangle$ $\quad \overrightarrow{BC} = \langle -2, 2, 4\rangle$

Since $\overrightarrow{BC} = 2\overrightarrow{AB} \Rightarrow \overrightarrow{BC} \parallel \overrightarrow{AB}$ $\therefore$ ABC are colinear //.

Alternatively, we could verify $\overrightarrow{AB} \times \overrightarrow{BC} = \langle 0,0,0\rangle$ $\therefore \overrightarrow{AB} \parallel \overrightarrow{BC}$ //.

Ex. Find equation of the line through $Q(3,1,-2)$ and is $\perp$ to the line L: $x = -1+t$, $y = -2+t$, $z = -1+t$, $-\infty < t < \infty$.



Let $P(-1+t, -2+t, -1+t)$ be any arbitrary pt. on L, then

$$d^2 = (-1+t-3)^2 + (-2+t-1)^2 + (-1+t+2)^2$$
$$= (t-4)^2 + (t-3)^2 + (t+1)^2$$
$$= t^2 - 8t + 16 + t^2 - 6t + 9 + t^2 + 2t + 1$$
$$= 3t^2 - 12t + 26$$

d^2 is the shortest when $t = \frac{12}{6} = 2$

Thus $P = (1, 0, 1)$ is the pt. where PQ is $\perp$ to L.

Now take $\overrightarrow{PQ} = \langle 2, 1, -3\rangle$ to be a vector $\parallel$ to the line,

$(3,1,2)$ as our reference pt., equations of the line are given by

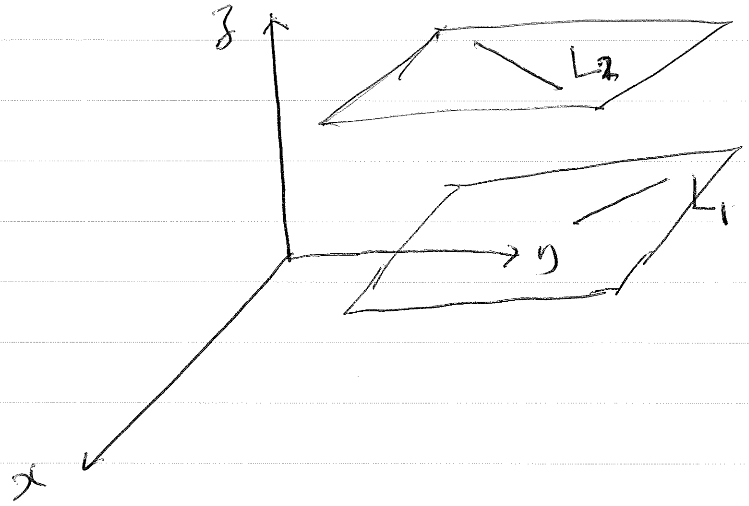
$$\begin{cases} x = 3 + 2t \\ y = 1 + t \\ z = -2 - 3t \end{cases} \quad -\infty < t < \infty$$

//.

L in its symmetric form is given by $\boxed{\dfrac{x-3}{2} = \dfrac{y-1}{1} = \dfrac{z+2}{-3}}$ //.

The Skew Lines in Space

In the $\mathbb{R}^2$ or 2-D setting, two lines L_1 & L_2 either intersect at one pt., parallel to eachother or co-incide. But in $\mathbb{R}^3$ or 3-D setting, there is a 3^{rd} possibility, when L_1 and L_2 are not parallel, but they belong to two separate $\parallel$ planes.



In this situation, L_1 & L_2 are known as a pair of skew lines.

Ex. Determine whether $L_1: x = 6+2t,\ y = 5+2t,\ z = 7+3t$ and $L_2: x = 7+3s,\ y = 5+3s,\ z = 10+5s$ are $\parallel$, intersect at one pt, co-incide or skew.

A systematic procedure:

(i) Find their direction vectors & see if their direction vectors are $\parallel$.
We have $L_1 \parallel \langle 2,2,3\rangle$ & $L_2 \parallel \langle 3,3,5\rangle$ $\because$ $\langle 2,2,3\rangle$ and $\langle 3,3,5\rangle$ not $\parallel$.
Therefore L_1, L_2 not $\parallel$.

(ii) Solving the 2 sets of equations for L_1 & L_2 by equating the x-y-z co-ordinates. there exists exactly one solution for (s,t), L_1 & L_2 intersect at one. If they have infinitely many solutions, they co-incide. Finally, if the 2 sets of equations are inconsistent, L_1 & L_2 are skew lines.

Indeed by equating the x, y & the z co-ordinates, we come up with

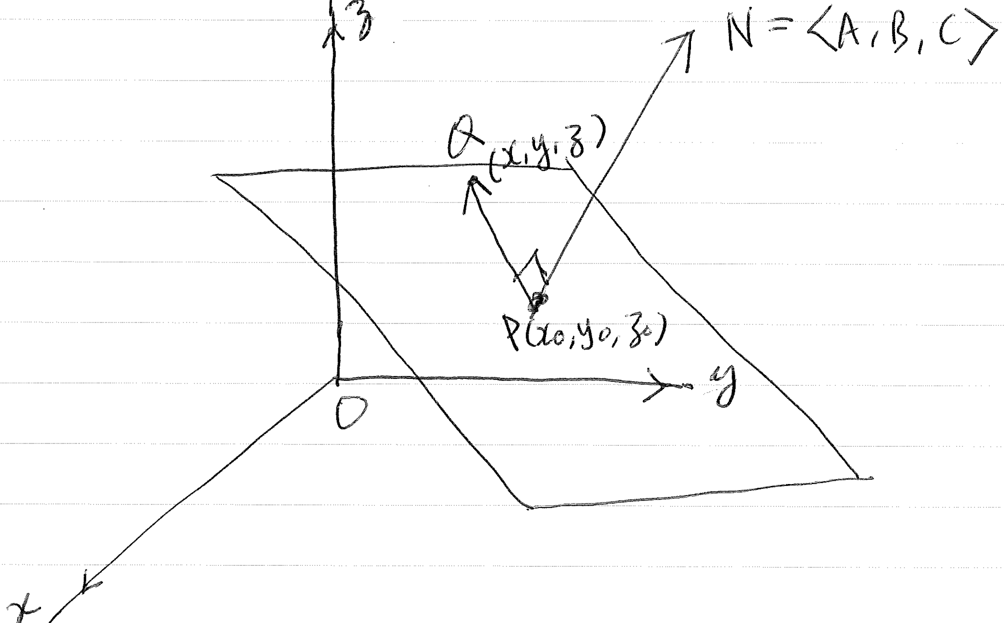
$$6+2t = 7+3s \quad \text{------ (i)}$$
$$5+2t = 5+3s \quad \text{------ (ii)}$$
$$7+3t = 10+5s \quad \text{------ (iii)}$$

(i) $\Rightarrow$ $2t-3s = 1$
(ii) $\Rightarrow$ $2t-3s = 0$
(iii) $\Rightarrow$ $3t-5s = 3$

(i) & (ii) are inconsistent, therefore there is no solution & L_1 & L_2 are skew //.

Equation of planes in Space

Just like a line is determined completely by a reference pt. on the line and a direction vector which is // to the Line. In the case of a plane in space, a reference point on the plane and a normal vector to the plane would determine the plane completely.



Let $Q(x, y, z)$ be any arbitrary point on the plane, then vector $\vec{PQ} = \langle x-x_0, y-y_0, z-z_0 \rangle$ is $\perp$ $\vec{N} = \langle A, B, C \rangle$

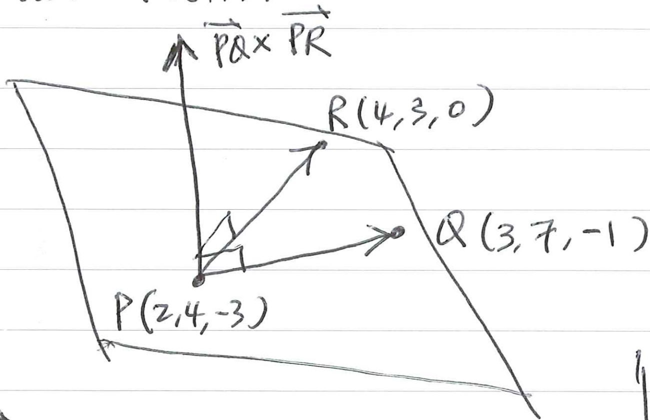
Hence $\vec{PQ} \cdot \vec{N} = 0 \Rightarrow$

$$\boxed{A(x-x_0) + B(y-y_0) + C(z-z_0) = 0}$$

$\leftarrow$ equation of plane passing thru (x_0, y_0, z_0) with Normal vector $\langle A, B, C \rangle$

Just like 2 pts on a line determines the line, 3 points in a plane determines the plane.

Ex Find the plane which passes through the 3 points $P(2,4,-3)$, $Q(3,7,-1)$ and $R(4,3,0)$



$\vec{PQ} = \langle 1,3,2\rangle$

$\vec{PR} = \langle 2,-1,3\rangle$

$$\vec{PQ}\times\vec{PR} = \langle 1,3,2\rangle \times \langle 2,-1,3\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 2 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= \left\langle \begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \right\rangle = \langle 11, 1, -7\rangle$$

(Note that the picture is wrong :) (How do we know?)

Taking $\vec{N} = \langle 11, 1, -7\rangle$ as our normal vector & $(2,4,-3)$ our reference point. Equation of the plane is given by,

$$11(x-2) + (1)(y-4) - 7(z+3) = 0 \quad //.$$

or $\quad 11x + y - 7z = 47$ //.

Note that, had we use $R(4,3,0)$ to be a reference pt. instead, we would come up with

$$11(x-4) + (y-3) - 7z = 0 \quad \text{or} \quad 11x + y - 7z = 47.$$

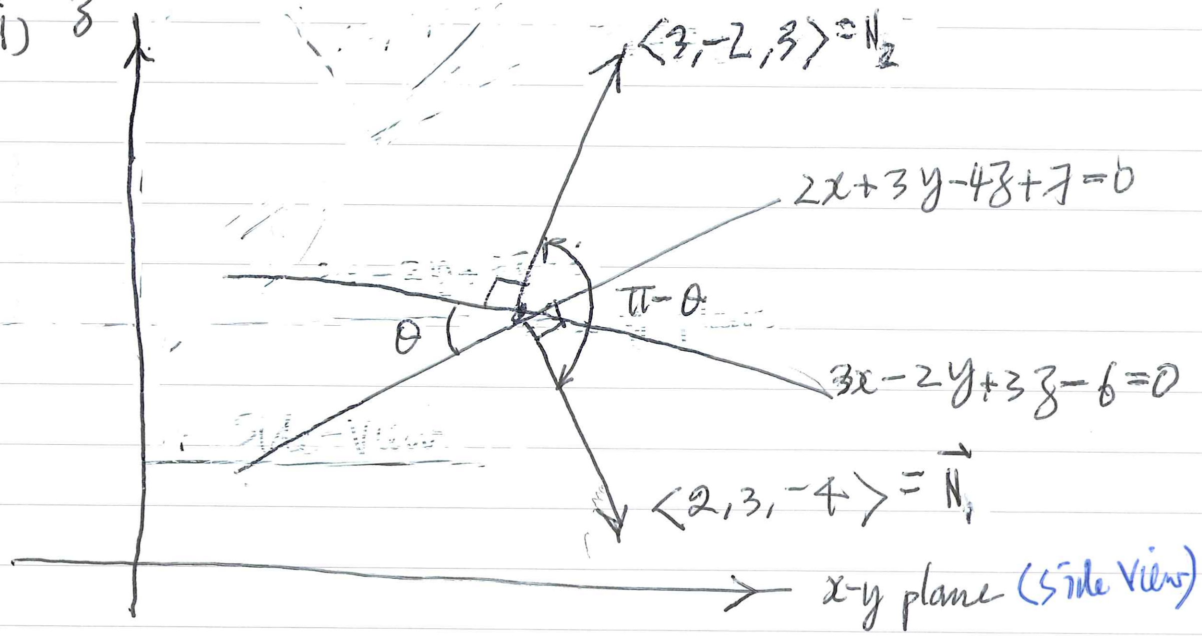
☆ Remark ☆ In general, whenever the equation of a plane is given in the form $Ax + By + Cz + D = 0$, $\vec{N} = \langle A, B, C\rangle$ is immediately a normal vector to the plane.

Ex. Given 2 planes $2x+3y-4z+7=0$ and $3x-2y+3z-6=0$ intersecting with each other, find

(i) The acute ∠ θ between the two planes

(ii) The equation for the line of intersection between the planes.

Solution: (i)



$$\frac{\langle 3,-2,3\rangle\cdot\langle 2,3,-4\rangle}{\sqrt{3^2+2^2+3^2}\,\sqrt{2^2+3^2+4^2}} = \frac{6-6-12}{\sqrt{22}\,\sqrt{29}} = \frac{-12}{\sqrt{638}}$$

$$\therefore \cos\theta = \frac{12}{\sqrt{638}} \quad (\because \theta \text{ is acute})$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{12}{\sqrt{638}}\right) \quad //$$

Remark: In general, if θ denotes the acute angle between the two planes, we could always have $\cos\theta = \dfrac{|\vec{N_1}\cdot\vec{N_2}|}{|\vec{N_1}||\vec{N_2}|}$ & hence

$$\boxed{\theta = \cos^{-1}\left(\frac{|\vec{N_1}\cdot\vec{N_2}|}{|\vec{N_1}||\vec{N_2}|}\right)}$$

where $\vec{N_1}$ and $\vec{N_2}$ are any two normal vectors to the planes respectively.

(ii) Let L be the line of intersection between the planes, L is co-
to both planes and is therefore $\perp$ to $\vec{N_1}$ and $\vec{N_2}$ simultaneously.
Hence, we could take $\vec{N_1} \times \vec{N_2}$ to be a direction vector $\parallel$ to L.

$$\vec{N_1} \times \vec{N_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -4 \\ 3 & -2 & 3 \end{vmatrix} = \left\langle \begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix}, -\begin{vmatrix} 2 & -4 \\ 3 & 3 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \right\rangle$$

$$= \langle 1, -18, -13 \rangle$$

It remains to get a reference pt. on L, we solve the equations of the planes simultaneously;

$$\begin{cases} 2x + 3y - 4z + 7 = 0 & \text{-----(i)} \\ 3x - 2y + 3z - 6 = 0 & \text{-----(ii)} \end{cases}$$

Setting $\boxed{z = 1}$, the system becomes

$$\begin{cases} 2x + 3y = -3 & \text{-----(i)} \\ 3x - 2y = 3 & \text{-----(ii)} \end{cases}$$

Now that $3 \times \text{(i)} - 2 \times \text{(ii)}$

$$\begin{aligned} 6x + 9y &= -9 \\ -)\ 6x - 4y &= 6 \\ \hline 13y &= -15 \end{aligned} \Rightarrow \boxed{y = \frac{-15}{13}}$$

Substitute backward, $\boxed{x = 3/13}$

Equation of L is therefore,

$$x = 3/13 + t,\quad y = \frac{-15}{13} - 18t,\quad z = 1 - 13t$$

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